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13.2 Projections of Nichols algebras.

Fix a Yetter-Drinfeld module $V \in {}^H_H\mathcal{YD}$ with subobj. $U, W \in {}^H_H\mathcal{YD}$.
$V = U \oplus W$. Thus

$$\begin{tikzcd} & & U \arrow[dl, "\subset"'] \arrow[d, "="] \\ W \subset & V \arrow[r, "\pi"'] & U \end{tikzcd}$$

$\pi : V = U \oplus W \longrightarrow U$
$\ker \pi = W$.

Lem 13.2.1. There is a unique Hopf alg map $\pi : A(V) \to A(U)$ which is identity on U, H, $\pi(W) = 0$. The map π is $\mathbb{N}_0$-graded w.r.t the grading given by
$\deg(H) = 0$, $\deg(U) = 0$, $\deg(W) = 1$.

and also w.r.t. the standard grading.
$\deg(V) = 1$, $\deg(H) = 0$.

proof. The alg $A(V) = B(V) \# H$ is generated by $\underset{v\#1}{V}$, $\underset{1\#h}{H}$.

This implies the uniqueness of π.
$\pi|_U = id$ $\pi|_H = id$ $\pi(W) = 0$.

On the other hand, $V \in {}^H_H\mathcal{YD}$ is $\mathbb{N}_0$-graded with $V(0) = U$, $V(1) = W$,
$V(n) = 0$ $n \geq 2$.

$\deg(H) = 0$

${}^H_H\mathcal{YD}(\Gamma - M_k)$

Then $B(V)$ is an $\mathbb{N}_0$-graded bialg. by Cor. 7.1.15 (1)

[$\Gamma \to \mathbb{N}_0$
Cor. 7.1.15. Let Γ abelian monoid, H. Γ-graded Hopf alg. bij. S.
W, V are Γ-graded obj in ${}^H_H\mathcal{YD}$.
(1) $B(V)$ is a Γ-graded Hopf alg quotient of $T(V)$ in ${}^H_H\mathcal{YD}$.

$V \in \Gamma - Gr\ {}^H_H\mathcal{YD}$ $V = \bigoplus V(g)$ $V(g) \subset V$ are subobj]

and $\underset{\shortparallel}{A}(U)$ is the degree 0 part of $A(V)$

and $\underset{\underset{\mathfrak{N}(U)\#H}{\|}}{A(U)}$ is the degree 0 part of $A(V)$

$\deg U = \deg(H) = 0$

Let $\pi : A(V) \longrightarrow A(U)$ be the graded projection

$$\left(\bigoplus_{n\geq 0} A(V)(n) \longrightarrow A(V)(0) \right)$$

Then π is a Hopf alg map. vanishing on W.

$$\left(H \longmapsto H(0) \quad \mathbb{N}_0\text{-graded Hopf alg map} \right)$$

and it is graded in the standard grading. #

Let $K = \{ x \in \underline{A(V)} \mid (id \otimes \pi)\underline{\Delta_{A(V)}}(x) = x \otimes 1_{A(U)} \}$ Hence

$$\begin{tikzcd} & A(U) \arrow[dl, "\subseteq"'] \arrow[d, "="] \\ K \subseteq A(V) \arrow[r, "\pi"'] & A(U) \end{tikzcd}$$

commutes, and

$K = A(V)^{co\, \underline{A(U)}}$

We view π as $\mathbb{N}_0$-graded map w.r.t the $\underline{\text{standard grading}}$ of $A(V)$ and $A(U)$. $\deg(V) = 1,\ \deg(H) = 0$

By $\underline{\text{Thm 5.5.6}}$, K is an $\underline{\mathbb{N}_0\text{-grading Hopf alg in } {}^{A(U)}_{A(U)}\mathcal{YD}}$

with grading $K(n) = A(V)(n) \cap K$. $\forall\, n \geq 0$. $K(n) \notin {}^{H}_{H}\mathcal{YD}$

[Thm 5.5.6. H Γ-graded. Hopf alg, bij. S.

(2) Let A Γ-graded Hopf alg, $\pi : A \to H$. $\gamma : H \to A$ $\overbrace{\text{graded map}}^{\text{Hopf alg}}$

....

Then R is a $\underline{\text{Hopf alg in } \mathcal{C} = {}^{H}_{H}\mathcal{YD}(\Gamma - Gr\, M_k)}$

with grading induced $\underline{R(\lambda) = R \cap A(\lambda)}$.]

with action coaction comultiplication

$ad : A(U) \otimes K \longrightarrow K \quad a \otimes x \longmapsto ad\, a(x) = a_1 x S(a_2)$

$\delta_K : K \longrightarrow A(U) \otimes K, \quad x \longmapsto (\pi \otimes id)\Delta_{A(\ldots)}(x)$

$ad: A(U)\otimes K \longrightarrow K \qquad a\otimes x \longmapsto ad\,a(x) = a_1 x S(a_2)$

$\delta_K: \quad K \to A(U)\otimes K, \quad x \mapsto (\pi\otimes id)\Delta_{A(U)}(x)$

$\Delta_K: K \to K\otimes K, \quad x \mapsto \vartheta_K(x_{(1)})\otimes x_{(2)}$

$\vartheta_K: A(V)\to K, \quad a \mapsto a_{(1)}\pi S(a_{(2)})$ [See Cor. 4.3.1]

The multiplication map

$$K \# A(U) \xrightarrow{\cong} A(V)$$

is $\mathbb{N}_0$-graded Hopf alg. isomorphism. [By Thm 5.5.6 (2)]

Denote the primitive elts of K by

$$P(K) = \{x \in K \mid \Delta_K(x) = x\otimes 1 + 1\otimes x\}$$

Lem 13.2.2 (1) $K = \{x \in B(V) \mid (id\otimes\pi)\Delta_{B(V)}(x) = x\otimes 1\}$

(2) $P(K) \subset K$ is $\mathbb{N}_0$-graded subobj. in ${}^{A(U)}_{A(U)}\mathcal{YD}$.

proof. (1) Let $\pi_H = \varepsilon\otimes id: A(U)\to H$ be the proj. onto H.

$\vartheta = id\otimes\varepsilon: A(V)\longrightarrow B(V)$.

If $x\in K$, then $(id\otimes\pi_H\pi)\Delta_{A(V)}(x) = x\otimes 1_H$

$\pi_H\pi: A(V)\longrightarrow H$

$\quad \| $

$\pi_{A(V)}$

hence $x\in A(V)^{coH} = B(V)$

and $x\otimes 1 = x^{(1)}x^{(2)}{}_{(-1)} \otimes \pi(x^{(2)}{}_{(0)})$

[$x\otimes 1 = x_{(1)}\otimes\pi_{A(V)}(x_{(2)})$

$= x_{(1)}\otimes\pi_{B(V)\#H}(x_{(2)})$

$= x^{(1)}x^{(2)}{}_{(-1)}\otimes\pi(x^{(2)}{}_{(0)})$

Since $x_{(1)}\otimes x_{(2)} = x^{(1)}(x^{(2)})_{(-1)}\otimes(x^{(2)})_{(0)}$.

Hence $\underset{\vartheta(x)\otimes 1}{x\otimes 1} = \vartheta(x^{(1)}x^{(2)}{}_{(-1)})\otimes\pi(x^{(2)}{}_{(0)}) = x^{(1)}\otimes\pi(x^{(2)})$

[$\vartheta(x^{(1)}x^{(2)}{}_{(-1)})\otimes\pi(x^{(2)}{}_{(0)}) = x^{(1)}\,\varepsilon((x^{(2)})_{(-1)})\otimes\pi(x^{(2)}{}_{(0)})$

$= (x^{(1)})\otimes\pi(x^{(2)})$]

Conversely, let $x\in B(V)$, $x^{(1)}\otimes\pi(x^{(2)}) = x\otimes 1$

Then $x\in K$, since $\pi: B(V)\to B(U)$ is left H-colinear ?

Conversely, let $x \in B(V)$, $x^{(-1)} \otimes \pi(x^{(0)}) = x \otimes 1$

Then $x \in K$, since $\pi : B(V) \to B(U)$ is left H-colinear.)

(2) follows from 5.5.2.

#

Let C be $\mathbb{N}_0$-graded coalg, X $\mathbb{N}_0$-graded left C-comodule. $\delta : X \to C \otimes X$ define δ_{ij}

$\delta_{ij} : X(i+j) \longrightarrow C(i) \otimes X(j)$ be the composition

$$X(i+j) \subset X \xrightarrow{\delta} C \otimes X \xrightarrow{\pi_i \otimes \pi_j} C(i) \otimes X(j)$$

We consider $\delta_{n-1,1}$. $\forall n \geq 1$.

prop. 13.2.3. Let C $\mathbb{N}_0$-graded coalg, X $\mathbb{N}_0$-graded left comod. Y a C-subcomodule of. Let $k \in \mathbb{Z}_{\geq 0}$ Assume $\delta_{n-k,k} : X(n) \longrightarrow C(n-k) \otimes X(k)$ is injective $\forall n \geq k$. Y is not contained $\bigoplus_{i=0}^{k-1} X(i)$ Then $Y \cap \bigoplus_{i=0}^{k} X(i) \neq 0$.

proof. By assumption. $0 \neq y = \sum_{i=0}^{n} x(i) \in Y$. $n \geq k$.

$x(i) \in X(i)$, $x(n) \neq 0$.

Let $\underset{\neq 0}{x} = x(n)$. $z = y - x = \sum_{i=0}^{n-1} x(i)$. Since $\delta_{n-k,k}$ is inj

$$0 \neq (\pi_{n-k} \otimes \pi_k)(\delta(x)) \in \underline{C(n-k)} \otimes X(k)$$

Then there exists $f \in C^*$ with $0 \neq \underline{\underline{f(x_{(-1)})x_{(0)}}} \in X(k)$ and $f(C(i)) = 0$ $\forall i \neq n-k$.

Note that $\underline{f(z_{(-1)})z_{(0)}} \in \bigoplus_{i=0}^{k-1} X(i)$

[$\delta(x_0) \in C(0) \otimes X(0)$, $\delta(x_1) \in C(0) \otimes X(1) + C(1) \otimes X(0)$

.... $\delta(x(n-1)) \in C(0) \otimes X(n-1) + C(1) \otimes X(n-2) + \cdots + C(n-1) \otimes X(0)$

$k=1$ $n-k = n-1$, $(f \otimes id)\delta(z) \in f(C(n-1))X(0) \subset X(0) = \bigoplus_{i=0}^{k-1} X(i)$.

$k=2$ $n-k = n-2$ $(f \otimes id)\delta(z) \in f(C(n-2)) \cancel{\otimes} (1)$

$+ \underset{\neq 0}{f(C(n-2))} \cancel{\otimes} (0) \subset \bigoplus_{i=0}^{1} X(i)$

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$f(z_{(n-1)})\, \cancel{x}_{(0)} \in \bigoplus_{i=0} X(i)$]

Thus $0 \neq f(y_{(-1)})\, y_{(0)} = \underbrace{f(x_{(-1)})\, \overset{\neq 0}{x_{(0)}}}_{\in X(k)} + \underbrace{f(z_{(-1)})\, z_{(0)}}_{\in \bigoplus_{i=0}^{n-1} X(i)} \in \bigoplus_{i=0}^{n} X(i)$

is a non-zero elt. in $Y \cap \bigoplus_{i=0}^{n} X(i)$. #